ISRAEL JOURNAL OF MATHEMATICS **158** (2007), 217–224
DOI: 10.1007/s11856-007-0011-0

ON SCATTERED EBERLEIN COMPACT SPACES

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ABSTRACT

We prove that scattered Eberlein compacta of Cantor–Bendixson height at most $\omega + 1$ are Uniform Eberlein compact spaces ($\omega + 1$ is optimal for this result). For a set X and $n \in \omega$, by $\sigma_n(2^X)$ we denote the subspace of the product 2^X consisting of all characteristic functions of sets of cardinality $\leq n$. We give an example of an Eberlein compactum K of weight ω_ω and of Cantor–Bendixson height 3 which cannot be embedded into any $\sigma_n(2^X)$.

Introduction

This paper is concerned with scattered Eberlein compact spaces. Let us recall that a space K is an **Eberlein** compact space if K is homeomorphic to a weakly compact subset of a Banach space. If K is homeomorphic to a weakly compact subset of a Hilbert space, then we say that K is a **Uniform Eberlein** compact

* Research of the first author supported by NSERC of Canada.
Murray Bell died on December 9, 2001.

** Research of the second author supported by KBN grants 2 P03A 011 15 and 2 P03A 004 23. The main part of this research was done while the second author was visiting the University of Manitoba in 2000. He expresses his gratitude to the Department of Mathematics of U.M. for its hospitality.
Received August 4, 2005 and in revised form December 14, 2005

space. For a scattered space K, by **Cantor–Bendixson height** of K we mean the minimal ordinal α such that the Cantor–Bendixson derivative $K^{(\alpha)}$ of the space K is empty. The Cantor–Bendixson height of a compact scattered space is always a nonlimit ordinal. Alster [2] has proved that all scattered **Corson** compacta (i.e., compact spaces embeddable in the Σ-products of real lines) are Eberlein compact. We prove that, under additional restriction of Cantor–Bendixson height of these spaces, they are in fact Uniform Eberlein:

THEOREM 0.1: *Every scattered Eberlein compact space of height less than or equal to $\omega + 1$ is Uniform Eberlein.*

We prove this theorem in Section 2.

For a set X and $n \in \omega$, by $\sigma_n(2^X)$ we denote the subspace of the product 2^X consisting of all characteristic functions of sets of cardinality $\leq n$. The space $\sigma_n(2^X)$ is Eberlein compact of height $n+1$ (for infinite X). Argyros and Godefroy have proved that such spaces are in some sense universal for Eberlein compacta of finite height and weight $< \omega_\omega$:

RESULT 0.2 (Argyros and Godefroy): *Every Eberlein compactum K of weight $< \omega_\omega$ and of finite height can be embedded into $\sigma_n(2^X)$ for some set X and $n \in \omega$.*

This (unpublished) result can be also derived from Theorem 4.8 from [7] and Theorem 1.1 from [10]. We give an example showing that the restriction on weight of K in the above theorem is necessary:

EXAMPLE 0.3: *There exists an Eberlein compactum K of weight ω_ω and of height 3 which cannot be embedded into any $\sigma_n(2^X)$.*

The construction of the example is given in Section 3. For this compactum K, the spaces $C(K)$ and $c_0(\omega_\omega)$ are examples of weakly compactly generated (WCG) Banach spaces which are Lipschitz isomorphic and not isomorphic; see [10]. This is related to some results of Argyros, Castillo, Granero, Jimenez, and Moreno, from the paper [3], which show that similar phenomena concerning the space $c_0(\Gamma)$ also appear at the cardinality ω_ω.

The space K from Example 0.3 indicates that scattered Eberlein compacta of height 3 may have quite complicated structure. In the paper [4] we investigate the problem of the existence of universal spaces (in the sense of either embeddings or continuous images) in some classes of scattered Eberlein compacta of height 3.

1. Notation and auxiliary results

For a set X and $n \in \omega$, we use the standard notation $[X]^n = \{A \subset X : |A| = n\}$, $[X]^{\leq n} = \bigcup\{[X]^k : k \leq n\}$ and $[X]^{<\omega} = \bigcup\{[X]^k : k < \omega\}$.

Lemma 1.1 (cf. [10, Lemma 2.2]): *Let K be a scattered Eberlein compactum. Then there exists a family $\{U_a : a \in K\}$ such that*

(a) *for every $a \in K$, U_a is a clopen neighborhood of a,*
(b) *for every ordinal α and $a \in K^{(\alpha)} \setminus K^{(\alpha+1)}$, $U_a \cap K^{(\alpha)} = \{a\}$,*
(c) *$\{U_a : a \in K\}$ is point-finite.*

Proof: Since K is a scattered Eberlein compactum, we can assume that K is a subset of $\{\chi_A \in 2^X : |A| < \omega\}$ for some set X; see [2]. For every $a = \chi_A \in K$ we put $V_a = \{\chi_B \in K : A \subset B\}$. It is clear that the family of clopen neighborhoods $\{V_a : a \in K\}$ is point-finite. For every ordinal α, each point $a \in K^{(\alpha)} \setminus K^{(\alpha+1)}$ is isolated in $K^{(\alpha)}$, therefore we can find a clopen neighborhood U_a of a such that $U_a \cap K^{(\alpha)} = \{a\}$. Obviously, we can require that $U_a \subset V_a$, hence the family $\{U_a : a \in K\}$ is point-finite. ∎

2. Eberlein compacta of height at most $\omega + 1$

By $\mathbb{N}$ we denote the set of positive integers.

Recall that a family $\mathcal{U}$ of subsets of a set X is point-n if each element of X belongs to at most n members of the family $\mathcal{U}$. We say that a family $\mathcal{U}$ is **σ-point-bounded** if $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$, where $\mathcal{U}_n$ is a point-n family for each $n \in \mathbb{N}$ (in another terminology, $\mathcal{U}$ is a countable union of subfamilies of finite order). It is clear that the countable union of σ-point-bounded families is also σ-point-bounded.

Let us recall the following characterization of Uniform Eberlein compact spaces (see [5, Theorem] and [6]):

Result 2.1 (Benyamini and Starbird): *A compact space K is Uniform Eberlein if and only if K has a T_0-separating σ-point-bounded family of cozero-sets.*

Lemma 2.2: *Let λ be a cardinal and m be a positive integer. Every family $\{U_\alpha : \alpha \in \lambda\}$ of subsets of a certain set satisfying the following condition*

$$(2.1) \qquad (\forall A \in [\lambda]^m)\left[\left|\left\{\beta \in \lambda : \bigcap\{U_\alpha : \alpha \in A\} \cap U_\beta \neq \emptyset\right\}\right| \leq \omega\right]$$

is σ-point-bounded.

Proof: We will prove the lemma by induction on λ. The case $\lambda \le \omega$ is trivial, we can write $\{U_\alpha : \alpha \in \lambda\}$ as a countable union of families $\{U_\alpha\}$ for $\alpha \in \lambda$.

Suppose that $\lambda > \omega$ and the assertion of our Lemma holds true for cardinals $\kappa < \lambda$. Consider the family $\{U_\alpha : \alpha \in \lambda\}$ satisfying condition (2.1) for a certain $m \ge 1$.

For every infinite set $X \subset \lambda$, we define

$$f_1(X) = X \cup \bigcup \{\{\beta \in \lambda : \bigcap \{U_\alpha : \alpha \in A\} \cap U_\beta \ne \emptyset\} : A \in [X]^m\}.$$

The condition (2.1) implies that $|f_1(X)| = |X|$. We define inductively $f_{n+1}(X) = f_1(f_n(X))$, for $n \ge 1$, and we put $g(X) = \bigcup\{f_n(X) : n \ge 1\}$ for every infinite $X \subset \lambda$. Again, we have $|g(X)| = |X|$. The definition of g easily implies the following property of the set $g(X)$:

$$(2.2) \qquad (\forall A \in [g(X)]^m)\,(\forall \beta \in (\lambda \setminus g(X)))\,\Big[\bigcap\{U_\alpha : \alpha \in A\} \cap U_\beta = \emptyset\Big].$$

Next, by transfinite induction we define sets $Y_\alpha \subset \lambda$ for $\alpha \in [\omega, \lambda)$. We put $Y_\omega = g(\omega)$ and $Y_\alpha = g(\bigcup\{Y_\beta : \beta \in [\omega, \alpha)\} \cup \alpha)$ for $\alpha \in (\omega, \lambda)$. One can easily compute that $|Y_\alpha| = |\alpha|$ for every α. It is also clear that $\bigcup\{Y_\alpha : \alpha \in [\omega, \lambda)\} = \lambda$ and $Y_\alpha \subset Y_\beta$ for $\alpha < \beta$. From property (2.2) it follows that

$$(2.3) \qquad (\forall A \in [Y_\alpha]^m)\,(\forall \beta \in (\lambda \setminus Y_\alpha))\Big[\bigcap\{U_\alpha : \alpha \in A\} \cap U_\beta = \emptyset\Big].$$

We take $Z_\alpha = Y_\alpha \setminus \bigcup\{Y_\beta : \beta < \alpha\}$ for $\alpha \in [\omega, \lambda)$. The sets Z_α are pairwise disjoint and have cardinalities less than λ. By the inductive assumption all families $\{U_\beta : \beta \in Z_\alpha\}$ are σ-point-bounded. For every $\alpha \in [\omega, \lambda)$, let $Z_\alpha = \bigcup\{Z_\alpha^n : n \in \mathbb{N}\}$ be a decomposition of Z_α such that the family $\{U_\beta : \beta \in Z_\alpha^n\}$ is point-n for each n. We define $Z^n = \bigcup\{Z_\alpha^n : \alpha \in [\omega, \lambda)\}$ for $n \ge 1$. To finish the proof it will be enough to show that, for every $n \in \mathbb{N}$, the family $\{U_\alpha : \alpha \in Z^n\}$ is point-$(n + m - 1)$. Take a finite set $A \subset Z^n$ such that $\bigcap\{U_\alpha : \alpha \in A\} \ne \emptyset$. Suppose that $|A| \ge m$ and define $\gamma = \min\{\beta \in [\omega, \lambda) : |A \cap Y_\beta| \ge m\}$. By condition (2.3) we infer that $A \subset Y_\gamma$. The choice of γ implies that $|A \setminus Z_\gamma| \le m - 1$. Since $A \subset Z^n$ we have $A \cap Z_\gamma = A \cap Z_\gamma^n$; hence $|A \cap Z_\gamma| \le n$. Finally, we obtain the required inequality $|A| = |A \cap Z_\gamma| + |A \setminus Z_\gamma| \le n + m - 1$. ∎

As an immediate consequence of the above lemma we obtain the following

COROLLARY 2.3: *Let $l \ge 1$ and let $\mathcal{U} = \{U_t : t \in T\}$ be a point-finite family such that the intersection $\bigcap\{U_t : t \in A\}$ is finite for every $A \in [T]^l$. Then the family $\mathcal{U}$ is σ-point-bounded.*

Proof of Theorem 0.1: Let $\{U_a : a \in K\}$ be the family of clopen neighborhoods given by Lemma 1.1. The condition (b) from Lemma 1.1 implies that the family $\{U_a : a \in K\}$ is T_0-separating. Hence, by Result 2.1 it suffices to show that $\{U_a : a \in K\}$ is σ-point-bounded. Since the height of K is at most $\omega+1$, we have $K = \bigcup\{K^{(n)} \setminus K^{(n+1)} : n \leq \omega\}$. Therefore, it is enough to prove that, for every $n \leq \omega$, the family $\{U_a : a \in K^{(n)} \setminus K^{(n+1)}\}$ is σ-point-bounded. We may assume that $n < \omega$, because the set $K^{(\omega)} \setminus K^{(\omega+1)}$ is finite. Fix $n < \omega$. We will choose inductively, for $k = 1, 2, \ldots, n$, the families $\{A(i_1, \ldots, i_k) : (i_1, \ldots, i_k) \in \mathbb{N}^k\}$ of subsets of $K^{(n)} \setminus K^{(n+1)}$ such that:

(i) $\bigcup\{A(i_1) : i_1 \in \mathbb{N}\} = K^{(n)} \setminus K^{(n+1)}$,

(ii) for every $(i_1, \ldots, i_{k-1}) \in \mathbb{N}^{k-1}$, we have

$$\bigcup\{A(i_1, \ldots, i_{k-1}, i_k) : i_k \in \mathbb{N}\} = A(i_1, \ldots, i_{k-1}),$$

(iii) for every $(i_1, \ldots, i_k) \in \mathbb{N}^k$, the family

$$\{U_a \cap (K^{(n-k)} \setminus K^{(n-k+1)}) : a \in A(i_1, \ldots, i_k)\} \text{ is point-}i_k.$$

We start with $k = 1$. For every distinct $a, b \in K^{(n)} \setminus K^{(n+1)}$, the clopen set $U_a \cap U_b$ is contained in $K \setminus K^{(n)}$ by condition (b) of Lemma 1.1. Therefore, the intersection $U_a \cap U_b \cap (K^{(n-1)} \setminus K^{(n)})$ is finite. By condition (c) of Lemma 1.1 we can apply Corollary 2.3, for the family $\{U_a \cap (K^{(n-1)} \setminus K^{(n)}) : a \in K^{(n)} \setminus K^{(n+1)}\}$ and $l = 2$, to conclude that this family is σ-point-bounded. Let $K^{(n)} \setminus K^{(n+1)} = \bigcup\{A(i_1) : i_1 \in \mathbb{N}\}$ be a decomposition of $K^{(n)} \setminus K^{(n+1)}$ such that the family $\{U_a \cap (K^{(n-1)} \setminus K^{(n)}) : a \in A(i_1)\}$ is point-i_1 for every $i_1 \in \mathbb{N}$.

Now, suppose that $k > 1$ and we have constructed, for $j < k$ and $(i_1, \ldots, i_j) \in \mathbb{N}^j$, the sets $A(i_1, \ldots, i_j)$ satisfying conditions (i)–(iii). Fix $(i_1, \ldots, i_{k-1}) \in \mathbb{N}^{k-1}$ and put $l = \max(i_1, \ldots, i_{k-1}) + 1$. The condition (b) of Lemma 1.1 implies that the intersection $\bigcap\{U_a : a \in A\}$ is contained in $K \setminus K^{(n)}$ for each $A \in [A(i_1, \ldots, i_{k-1})]^l$. Moreover, from conditions (ii) and (iii) it follows that $\bigcap\{U_a : a \in A\} \cap (K^{(n-j)} \setminus K^{(n-j+1)}) = \emptyset$ for $j = 1, \ldots, k-1$. Hence we have $\bigcap\{U_a : a \in A\} \subset (K \setminus K^{(n-k+1)})$, which implies that the set $\bigcap\{U_a : a \in A\} \cap (K^{(n-k)} \setminus K^{(n-k+1)})$ is finite. From Corollary 2.3, applied to the family $\{U_a \cap (K^{(n-k)} \setminus K^{(n-k+1)}) : a \in A(i_1, \ldots, i_{k-1})\}$, we infer that this family is σ-point-bounded. Therefore, we can find sets $A(i_1, \ldots, i_k)$, for $i_k \in \mathbb{N}$, satisfying conditions (ii) and (iii).

Finally, conditions (i) and (ii) imply that the family $\{U_a : a \in K^{(n)} \setminus K^{(n+1)}\}$ is the union of countably many families $\{U_a : a \in A(i_1, \ldots, i_n)\}$ for $(i_1, \ldots, i_n) \in \mathbb{N}^n$. Repeating the argument from the above inductive construction, one can easily verify that, for given $(i_1, \dots, i_n) \in \mathbb{N}^n$ and $s = \max(i_1, \dots, i_n)$, the family $\{U_a : a \in A(i_1, \dots, i_n)\}$ is point-s. Obviously, this shows that $\{U_a : a \in K^{(n)} \setminus K^{(n+1)}\}$ is σ-point-bounded. ∎

Let us point out that the restriction on the height of the space K in Theorem 0.1 cannot be improved. There exist scattered Eberlein compact spaces K of height $\omega + 2$ which are not Uniform Eberlein; see [5] and [9].

3. Embeddings into $\sigma_n(2^X)$

In this section we construct the space K announced in Example 0.3. Our construction is based on an idea from [8, p. 16]; see also [1, Lemma 2]. For a verification of the properties of the space K we need two auxiliary facts. We will use the following simple property of the space $\sigma_n(2^X)$:

PROPOSITION 3.1: *Let S be a subset of $\sigma_n(2^X)$ for some set X and $n \in \omega$. Then there exists a point-2^n family $\{V_s : s \in S\}$ of open subsets of S such that $s \in V_s$ for all $s \in S$.*

Proof: Clearly, it is enough to find the required family of neighborhoods for $S = \sigma_n(2^X)$. For $A \in [X]^{\leq n}$ we take the standard clopen neighborhood $V_{\chi_A} = \{\chi_B \in \sigma_n(2^X) : A \subset B\}$ of χ_A. It is obvious that the family $\{V_{\chi_A} : \chi_A \in \sigma_n(2^X)\}$ is point-2^n. ∎

LEMMA 3.2: *Let $n \in \omega$ and φ: $[\omega_n]^n \to [\omega_n]^{<\omega}$ be an arbitrary map. Then there exists $A \in [\omega_n]^{n+1}$ such that, for every $\alpha \in A$, we have $\alpha \notin \varphi(A \setminus \{\alpha\})$.*

Proof: We will prove the lemma by induction on n. For $n = 0$ we have $[\omega_0]^0 = \{\emptyset\}$, hence it is enough to take any $k \in \omega_0 \setminus \varphi(\emptyset)$ and put $A = \{k\}$.

Now, suppose that lemma holds true for $n \in \omega$ and consider φ: $[\omega_{n+1}]^{n+1} \to [\omega_{n+1}]^{<\omega}$. The union $\bigcup\{\varphi(B) : B \in [\omega_n]^{n+1}\}$ has cardinality at most ω_n, therefore we can find $\beta \in \omega_{n+1} \setminus (\bigcup\{\varphi(B) : B \in [\omega_n]^{n+1}\} \cup \omega_n)$. We define $\psi[\omega_n]^n \to [\omega_n]^{<\omega}$ by $\psi(B) = \varphi(B \cup \{\beta\}) \cap \omega_n$ for $B \in [\omega_n]^n$. By the inductive assumption, there exists $C \in [\omega_n]^{n+1}$ such that $\alpha \notin \psi(C \setminus \{\alpha\})$ for every $\alpha \in C$. One can easily verify that the set $A = C \cup \{\beta\}$ has the required property for the map φ. ∎

The construction of Example 0.3: Let $S_n = [\omega_n]^n$, for $n \geq 1$, and $S = \bigcup\{S_n : n \geq 1\}$. For $n \geq 1$, we put $\mathcal{A}_n = \{\emptyset\} \cup \{\{B\} : B \in S_n\} \cup \{T \subset S_n : |T| = n + 1$ and $|\bigcup T| = n + 1\}$. Observe that every set $T \subset S_n$ satisfying the conditions $|T| = n+1$ and $|\bigcup T| = n+1$ has the form $[A]^n = \{A \setminus \{\alpha\} : \alpha \in A\}$ for some $A \in [\omega_n]^{n+1}$. We define $\mathcal{A} = \bigcup\{\mathcal{A}_n : n \geq 1\}$ and $K = \{\chi_T : T \in \mathcal{A}\}$ considered as a subspace of the product 2^S. A routine verification shows that K is a compact subset of 2^S of height 3 and weight equal to $|S| = \omega_\omega$. It remains to prove that K cannot be embedded into any $\sigma_n(2^X)$.

We consider the following closed subsets $K_n = \{\chi_T : T \in \mathcal{A}_n\}$ of K for $n \geq 1$. We will demonstrate the required property of K by showing that, for every $n \geq 1$, the space K_{2^n} admits no embedding into $\sigma_n(2^X)$. By Proposition 3.1 it is enough to prove that, for $n \geq 1$, every family of open sets $\{V_a : a \in K_n\}$ satisfying $a \in V_a$, for all $a \in K_n$, is not point-n. Given $B \in S_n$, we consider the standard clopen neighborhood $U_B = \{\chi_T \in K_n : B \in T\}$ of $\chi_{\{B\}}$. The point $\chi_{\{B\}}$ is the unique accumulation point of U_B. Therefore, every neighborhood $V_{\chi_{\{B\}}}$ of $\chi_{\{B\}}$ contains a set of the form $W_B = U_B \setminus F_B$, where F_B is a finite subset of $U_B \setminus \{\chi_{\{B\}}\}$. Hence, it is enough to show that the family $\{W_B : B \in S_n\}$ is not point-n. For every $\chi_T \in U_B \setminus \{\chi_{\{B\}}\}$ we have $T = [B \cup \{\alpha\}]^n$ for some $\alpha \in (\omega_n \setminus B)$. For $B \in S_n = [\omega_n]^n$ we put $\varphi(B) = \{\alpha \in \omega_n : \chi_{[B\cup\{\alpha\}]^n} \in F_B\}$. By Lemma 3.2 there exists $A \in [\omega_n]^{n+1}$ such that, for every $\alpha \in A$, we have $\alpha \notin \varphi(A \setminus \{\alpha\})$. Then, for $T = [A]^n$, the point χ_T belongs to $n+1$ sets $W_{A\setminus\{\alpha\}}$ for $\alpha \in A$. ∎

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